



NEW DIFFERENTIABILITY CONCEPTS FOR FUZZY DIFFERENTIAL EQUATION

B. Usha¹

¹ Department of Mathematics, Kongu Engineering College, Perundurai, Erode-638 060, Tamilnadu, India.

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Corresponding Author:

B. Usha

Abstract:

In this paper, numerical algorithms for solving “fuzzy ordinary differential equations” are considered. A scheme based on the Runge-Kutta method of order four is discussed in detail and this is followed by a complete error analysis.

Keywords: Fuzzy differential equations, Runge-Kutta method of order four, Trapezoidal fuzzy number

1. Introduction

Fuzzy differential equations are important tools to deal with dynamic systems in fuzzy environments. The concept of a fuzzy derivative was first introduced by S. L. Chang and L. A. Zadeh in [3]. Fuzzy differential equations and initial value problems were regularly treated by O.Kaleva in [6] and [7], S.Seikkala in [9]. The organized paper is as follows: In the first four sections, we recall some concepts, algorithm and introductory materials to deal with the fuzzy initial value problem. Solving numerically the fuzzy differential equation by Runge-Kutta method of order four is discussed in section 5. The proposed algorithm is illustrated by an example in the last section.

2. Preliminary

A trapezoidal fuzzy number u is defined by four real numbers $k < \ell < m < n$, where the base of the trapezoidal is the interval $[k, n]$ and its vertices at $x = \ell$, $x = m$. Trapezoidal fuzzy number will be written as $u = (k, \ell, m, n)$. The membership function for the trapezoidal fuzzy number $u = (k, \ell, m, n)$ is defined as the following :

$$u(x) = \begin{cases} \frac{x - k}{\ell - k}, & k \leq x \leq \ell \\ 1, & \ell \leq x \leq m \\ \frac{x - n}{m - n}, & m \leq x \leq n \end{cases} \quad (1)$$

we will have :

$$u > 0 \text{ if } k > 0; \quad u > 0 \text{ if } \ell > 0; \quad u > 0 \text{ if } m > 0; \quad \& \quad u > 0 \text{ if } n > 0.$$

Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \rightarrow [0,1]$) satisfying the following properties:

- (i) $\forall u \in R_F$, u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$;
- (ii) $\forall u \in R_F$, u is convex fuzzy set (i.e. $u(tx + (1 - t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0, 1], x, y \in R$);
- (iii) $\forall u \in R_F$, u is upper semi continuous on R ;
- (iv) $\overline{\{x \in R; u(x) > 0\}}$ is compact, where \bar{A} denotes the closure of A .

Then R_F is called the space of fuzzy numbers (see e.g. [7]). Obviously $R \subset R_F$. Here $R \subset R_F$ is understood as $R = \{\mathcal{X}_{\{x\}}; x \text{ is usual real number}\}$. We define the r-level set, $x \in R$:

$$[u]_r = \{x \mid u(x) \geq r\}, \quad 0 \leq r \leq 1; \quad (2)$$

Clearly, $[u]_0 = \{x \mid u(x) > 0\}$ is compact,

which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \bar{u}(r)]$. It is clear that the following statements are true,

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over $[0, 1]$,
2. $\bar{u}(r)$ is a bounded right continuous non increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \bar{u}(r)$ for all $r \in (0, 1]$,

for more details see [1],[2].

Let $D: R_F \times R_F \rightarrow R_+ \cup \{0\}$, $D(u, v) = \sup_{r \in [0,1]} \max \{ |\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)| \}$, be Hausdorff distance between fuzzy numbers, where $[u]_r = [\underline{u}(r), \bar{u}(r)]$, $[v]_r = [\underline{v}(r), \bar{v}(r)]$. The following properties are well-known (see e.g. [8]):

$$D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in R_F,$$

$$D(k.u, k.v) = |k|D(u, v), \quad \forall k \in R, u, v \in R_F,$$

$$D(u + v, w + e) \leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in R_F$$

and (R_F, D) is a complete metric space.

Theorem 2.1

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F . Then for arbitrarily fixed r , $0 \leq r \leq 1$, the numerical solutions of $\underline{y}(t_{n+1}; r)$ and $\bar{y}(t_{n+1}; r)$ converge to the exact solutions $\underline{Y}(t; r)$ and $\bar{Y}(t; r)$ uniformly in t .

Theorem 2.2

Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \leq r \leq 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n; r)$ and $\bar{y}^{(j)}(t_n; r)$ converge to the numerical solutions $\underline{y}(t_n; r)$ and $\bar{y}(t_n; r)$ in $t_0 \leq t_n \leq t_N$, when $j \rightarrow \infty$.

3 Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \quad (3)$$

where y is a fuzzy function of t , $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y , y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a trapezoidal or a trapezoidal shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \bar{y}]$. It means that the r -level set of $y(t)$ for

$t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)], \quad [y(t_0)]_r = [\underline{y}(t_0; r), \bar{y}(t_0; r)], \quad r \in (0, 1]$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$$\underline{f}(t, y) = F[t, \underline{y}, \overline{y}], \quad \overline{f}(t, y) = G[t, \underline{y}, \overline{y}].$$

Because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \overline{y}(t; r)] \quad (4)$$

$$\overline{f}(t, y(t); r) = G[t, \underline{y}(t; r), \overline{y}(t; r)] \quad (5)$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\}, \quad s \in R \quad (6)$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \quad r \in (0, 1], \quad (7)$$

where $\underline{f}(t, y(t); r) = \min \{f(t, u) | u \in [y(t)]_r\}$ (8)

$$\overline{f}(t, y(t); r) = \max \{f(t, u) | u \in [y(t)]_r\}. \quad (9)$$

Definition 3.1 A function $f: R \rightarrow R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\epsilon > 0, \delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \epsilon \quad \text{exists.}$$

Throughout this paper we also consider fuzzy functions which are continuous in metric D . Then the continuity of $f(t, y(t); r)$ guarantees the existence of the definition of $f(t, y(t); r)$ for $t \in [t_0, T]$ and $r \in [0, 1]$ [5]. Therefore, the functions G and F can be definite too.

4 Runge-Kutta method of order four

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \quad (10)$$

Assuming the following Runge-Kutta method with four slopes

$$y(t_{n+1}) = y(t_n) + W_1 K_1 + W_2 K_2 + W_3 K_3 + W_4 K_4 \quad (11)$$

where

$$\begin{aligned}
 K_1 &= hf(t_n, y(t_n)) \\
 K_2 &= hf(t_n + c_2h, y(t_n) + a_{21}K_1) \\
 K_3 &= hf(t_n + c_3h, y(t_n) + a_{31}K_1 + a_{32}K_2) \\
 K_4 &= hf(t_n + c_4h, y(t_n) + a_{41}K_1 + a_{42}K_2 + a_{43}K_3)
 \end{aligned}$$

and the parameters $c_2, c_3, c_4, a_{21}, \dots, a_{43}$ and W_1, \dots, W_4 are chosen to make y_{n+1} closer to $y(t_{n+1})$. There are thirteen parameters to be determined. Now, Taylor's series expansion about t_n gives

$$\begin{aligned}
 y(t_{n+1}) &= y(t_n) + \frac{h}{1!} y'(t_n) + \frac{h^2}{2!} y''(t_n) + \frac{h^3}{3!} y'''(t_n) + \frac{h^4}{4!} y^{(4)}(t_n) + \dots \\
 &= y(t_n) + \frac{h}{1!} f(t_n, y(t_n)) + \frac{h^2}{2!} [f_t + ff_y]_{t_n} + \frac{h^3}{3!} [f_{tt} + 2ff_{ty} + f^2 f_{yy} + f_y(f_t + ff_y)]_{t_n} + \dots
 \end{aligned}
 \tag{12}$$

If we set

$$\begin{aligned}
 K_1 &= hf_n \\
 K_2 &= hf(t_n + c_2h, y(t_n) + a_{21}K_1) \\
 &= h \left\{ f_n + \frac{h}{1!} [c_2 f_t + a_{21} ff_y]_{t_n} \right. \\
 &\quad \left. + \frac{h^2}{2!} \left[c_2^2 f_{tt} + 2c_2 a_{21} ff_{ty} \right]_{t_n} + \frac{h^3}{3!} \left[c_2^3 f_{ttt} + a_{21}^3 f^3 f_{yyy} \right. \right. \\
 &\quad \left. \left. + 3c_2^2 a_{21} ff_{t^2y} + 3c_2 a_{21}^2 f^2 f_{tyy} \right]_{t_n} + \dots \right\}
 \end{aligned}$$

$$K_3 = hf(t_n + c_3h, y(t_n) + a_{31}K_1 + a_{32}K_2)$$

$$\begin{aligned}
 & \left[f_n + c_3 h f_t + a_{31} h f f_y + a_{32} \left[h f + h^2 (c_2 f_t + a_{21} f f_y) + \frac{h^3}{2} \left(\begin{aligned} & c_2^2 f_{tt} + 2c_2 a_{21} f f_{ty} \\ & + a_{21}^2 f^2 f_{yy} \end{aligned} \right) \right] f_y \right] \\
 = h & \left\{ + \frac{1}{2!} \left[\begin{aligned} & c_3^2 h^2 f_{tt} + 2c_3 h (a_{31} + a_{32}) h f + a_{32} h^2 (c_2 f_t + a_{21} f f_y) f_{ty} \\ & + (a_{31}^2 + a_{32}^2) h^2 f^2 f_{yy} + 2a_{31} a_{32} h f f_{yy} \\ & (h f + h^2 (c_2 f_t + a_{21} f f_y)) f_{yy} \end{aligned} \right] \right. \\
 & \left. + \frac{1}{3!} \left[\begin{aligned} & c_3^3 h^3 f_{ttt} + (a_{31}^3 + a_{32}^3 + 3a_{31}^2 a_{32} + 3a_{31} a_{32}^2) h^3 f^3 f_{yyy} \\ & + 3c_3 h^3 f^2 f_{tyy} (a_{31}^2 + a_{32}^2 + 2a_{31} a_{32}) + 3c_3^2 h^3 (a_{31} + a_{32}) f f_{ty} \end{aligned} \right] + \dots \right\} \\
 & \left[f_n + \frac{h}{1!} [c_3 f_t + (a_{31} + a_{32}) f_n f_y]_{t_n} + \frac{h^2}{2!} \left[\begin{aligned} & 2a_{32} (c_2 f_t + a_{21} f f_y) f_y + c_3^2 f_{tt} \\ & + 2c_3 f_n (a_{31} + a_{32}) + 2a_{31} a_{32} f^2 \\ & + (a_{31}^2 + a_{32}^2) f_n^2 f_{yy} \end{aligned} \right]_{t_n} \right] \\
 = h & \left\{ + \frac{h^3}{3!} \left[\begin{aligned} & 3(c_2^2 f_{tt} + 2c_2 a_{21} f f_{ty} + a_{21}^2 f^2 f_{yy}) a_{32} f_y + 6c_3 a_{32} (c_2 f_t + a_{21} f f_y) f_{ty} \\ & + 6a_{31} a_{32} f_n f_{yy} (c_2 f_t + a_{21} f f_y) f_{yy} + c_3^3 f_{ttt} \\ & + (a_{31}^3 + a_{32}^3 + 3a_{31} a_{32}^2 + 3a_{31}^2 a_{32}) f^3 f_{yyy} \\ & + 3c_3 f^2 f_{tyy} (a_{31}^2 + a_{32}^2 + 2a_{31} a_{32}) + 3c_3^2 f (a_{31} + a_{32}) f_{ty} \end{aligned} \right]_{t_n} \right\} \\
 & + \dots
 \end{aligned}$$

$$K_4 = h f(t_n + c_4 h, y(t_n) + a_{41} K_1 + a_{42} K_2 + a_{43} K_3)$$

$$=h \left\{ \begin{aligned} & \left[f_n + c_4 h f_t + a_{41} h f + a_{42} \left(h f + \frac{h^2}{1!} (c_2 f_t + a_{21} f f_y) + \frac{h^3}{2!} (c_2^2 f_{tt} + 2c_2 a_{21} f f_{ty}) + a_{21}^2 f^2 f_{yy} \right) + a_{43} \left(\frac{h^3}{2} \left(\begin{aligned} & (c_3 f_t + f f_y (a_{31} + a_{32})) + \right. \right. \right. \\ & \left. \left. \left. \begin{aligned} & 2a_{32} (c_2 f_t + a_{21} f f_y) f_y \\ & + c_3^2 f_{tt} + 2c_3 f_n (a_{31} + a_{32}) \\ & + 2a_{31} a_{32} f^2 + (a_{31}^2 + a_{32}^2) \end{aligned} \right) \right. \right. \\ & \left. \left. \left. f_n^2 f_{yy} \right) \right) \right] \\ & \left[f_y + 2c_4 h f_{ty} + \left(a_{41} h f + a_{42} \left(h f + \frac{h^2}{1!} (c_2 f_t + a_{21} f f_y) + \frac{h^3}{2} (c_2^2 f_{tt} + 2c_2 a_{21} f f_{ty} + a_{21}^2 f^2 f_{yy}) \right) \right) f_{yy} + \frac{1}{2!} (c_4^2 h^2 f_{tt}) + \dots \right. \\ & \left. \left. + a_{43} (h f + h^2 (c_3 f_t + f f_y (a_{31} + a_{32}))) \right] \right\} \end{aligned} \right.$$

$$=h \left\{ \begin{aligned} & \left[f_n + c_4 h f_t + \frac{1}{2!} (c_4^2 h^2 f_{tt}) + \left[\frac{h}{1!} [(a_{41} + a_{42} + a_{43}) f]_{t_n} + \frac{h^2}{2!} [2a_{42} (c_2 f_t + a_{21} f f_y) + 2a_{43} (c_3 f_t + (a_{31} + a_{32}) f f_y)]_{t_n} \right. \right. \\ & \left. \left. + \frac{h^3}{3!} \left[3a_{42} \left(\begin{aligned} & c_2^2 f_{tt} \\ & + 2c_2 a_{21} f f_{ty} \\ & + a_{21}^2 f^2 f_{yy} \end{aligned} \right) + 3a_{43} \left(\begin{aligned} & 2(c_2 f_t + a_{21} f f_y) a_{32} f_y + c_3^2 f_{tt} \\ & + 2c_3 f (a_{31} + a_{32}) \\ & + (a_{31}^2 + a_{32}^2) f_n^2 f_{yy} + 2a_{31} a_{32} f^2 \end{aligned} \right) \right]_{t_n} \right] \\ & \left[f_y + \frac{h}{2} [4c_4 f_{ty} + 2f f_{yy} (a_{41} + a_{42} + a_{43})]_{t_n} + h^2 [a_{42} f_{yy} (c_2 f_t + a_{21} f f_y) + a_{43} f_{yy} (c_3 f_t + f f_y (a_{31} + a_{32}))]_{t_n} \right] \\ & \left. + h^3 \left[\frac{a_{42} f_{yy}}{2} \left(\begin{aligned} & c_2^2 f_{tt} + 2c_2 a_{21} f f_{ty} \\ & + a_{21}^2 f^2 f_{yy} \end{aligned} \right) + \frac{a_{43} f_{yy}}{2} \left(\begin{aligned} & 2a_{32} f_y (c_2 f_t + a_{21} f f_y) f_y + c_3^2 f_{tt} + 2c_3 a_{31} f \\ & + 2c_3 a_{32} f f_y + (a_{31}^2 + a_{32}^2) f_n^2 f_{yy} + 2a_{31} a_{32} f^2 \end{aligned} \right) \right]_{t_n} \right] \right\} + \end{aligned} \right.$$

$$= h \left\{ \begin{aligned} & \left[f_n + \frac{h}{1!} [c_4 f_t + ff_y (a_{41} + a_{42} + a_{43})]_{t_n} + \frac{h^2}{2!} \left[c_4^2 f_{tt} + f(a_{41} + a_{42} + a_{43}) (4c_4 f_{ty} + 2ff_{yy} (a_{41} + a_{42} + a_{43})) \right. \right. \\ & \left. \left. + 2a_{42} f_y (c_2 f_t + a_{21} ff_y) + 2a_{43} f_y (c_3 f_t + ff_y (a_{31} + a_{32})) \right]_{t_n} \right] \\ & \left. + \frac{h^3}{3!} \left[\frac{3}{2} \left[2a_{42} (c_2 f_t + a_{21} ff_y) + 2a_{43} (c_3 f_t + (a_{31} + a_{32}) ff_y) \right] [4c_4 f_{ty} + 2ff_{yy} (a_{41} + a_{42} + a_{43})] + \right. \right. \\ & \left. \left. \left[\begin{aligned} & 3a_{42} \left(\begin{aligned} & c_2^2 f_{tt} + \\ & 2c_2 a_{21} ff_{ty} + \\ & a_{21}^2 f^2 f_{yy} \end{aligned} \right) + 3a_{43} \left(\begin{aligned} & 2(c_2 f_t + a_{21} ff_y) a_{32} f_y + c_3^2 f_{tt} \\ & + 2c_3 f (a_{31} + a_{32}) \\ & + (a_{31}^2 + a_{32}^2) f_n^2 f_{yy} + 2a_{31} a_{32} f^2 \end{aligned} \right) \right] f_y \end{aligned} \right]_{t_n} \right\} \text{Substituting}$$

the values of K_1, K_2, K_3 & K_4 in (11), we get

$$y(t_{n+1}) = y(t_n) + [W_1 + W_2 + W_3 + W_4] h f_n + \frac{h^2}{1!} \left[\begin{aligned} & W_2 (c_2 f_t + a_{21} ff_y) + W_3 (c_3 f_t + (a_{31} + a_{32}) f_n f_y) \\ & + W_4 (c_4 f_t + ff_y (a_{41} + a_{42} + a_{43})) \end{aligned} \right]_{t_n}$$

$$+ \frac{h^3}{2!} \left[\begin{aligned} & W_2 (c_2^2 f_{tt} + 2c_2 a_{21} ff_{ty} + a_{21}^2 f^2 f_{yy}) + W_3 \left(2(c_2 f_t + a_{21} ff_y) a_{32} f_y + c_3^2 f_{tt} + 2c_3 a_{31} f_n + \right. \\ & \left. 2c_3 a_{32} f_n f_y + (a_{31}^2 + a_{32}^2 + 2a_{31} a_{32}) f_n^2 f_{yy} \right) \\ & + W_4 \left(c_4^2 f_{tt} + f(a_{41} + a_{42} + a_{43}) (4c_4 f_{ty} + 2ff_{yy} (a_{41} + a_{42} + a_{43})) \right. \\ & \left. + 2a_{42} f_y (c_2 f_t + a_{21} ff_y) + 2a_{43} f_y (c_3 f_t + ff_y (a_{31} + a_{32})) \right) \end{aligned} \right]_{t_n}$$

$$+ \frac{h^4}{3!} \left[\begin{aligned} & W_2 \left[\begin{aligned} & c_2^3 f_{ttt} + a_{21}^3 f^3 f_{yyy} \\ & + 3c_2^2 a_{21} ff_{ty} \\ & + 3c_2 a_{21}^2 f^2 f_{yy} \end{aligned} \right] + W_3 \left[\begin{aligned} & 3(c_2^2 f_{tt} + 2c_2 a_{21} ff_{ty} + a_{21}^2 f^2 f_{yy}) a_{32} f_y \\ & + 6c_3 a_{32} (c_2 f_t + a_{21} ff_y) f_{ty} + 6a_{31} a_{32} f_n f_{yy} (c_2 f_t + a_{21} ff_y) f_{yy} \\ & + c_3^3 f_{ttt} + (a_{31}^3 + a_{32}^3 + 3a_{31} a_{32}^2 + 3a_{31}^2 a_{32}) f^3 f_{yyy} \\ & + 3c_3 f^2 f_{yy} (a_{31}^2 + a_{32}^2 + 2a_{31} a_{32}) + 3c_3^2 f (a_{31} + a_{32}) f_{ty} \end{aligned} \right] \\ & + W_4 \left[\begin{aligned} & f(a_{41} + a_{42} + a_{43}) (6a_{42} f_{yy} (c_2 f_t + a_{21} ff_y) + 6a_{43} f_{yy} (c_3 f_t + ff_y (a_{31} + a_{32}))) \\ & + \frac{3}{2} \left[2a_{42} (c_2 f_t + a_{21} ff_y) + 2a_{43} (c_3 f_t + (a_{31} + a_{32}) ff_y) \right] [4c_4 f_{ty} + 2ff_{yy} (a_{41} + a_{42} + a_{43})] \\ & + \left[\begin{aligned} & 3a_{42} \left(\begin{aligned} & c_2^2 f_{tt} + \\ & 2c_2 a_{21} ff_{ty} + \\ & a_{21}^2 f^2 f_{yy} \end{aligned} \right) + 3a_{43} \left(\begin{aligned} & 2(c_2 f_t + a_{21} ff_y) a_{32} f_y + c_3^2 f_{tt} \\ & + 2c_3 f (a_{31} + a_{32}) \\ & + (a_{31}^2 + a_{32}^2) f_n^2 f_{yy} + 2a_{31} a_{32} f^2 \end{aligned} \right) \right] f_y \end{aligned} \right]_{t_n} + \dots \quad (13)$$

Comparing the coefficients of h, h^2, h^3 & h^4 in (12) & (13), we obtain

$$\begin{aligned}
 c_2 &= a_{21}, \quad c_3 = a_{31} + a_{32}, \quad c_4 = a_{41} + a_{42} + a_{43}, \\
 W_1 + W_2 + W_3 + W_4 &= 1, \quad W_2 c_2 + W_3 c_3 + W_4 c_4 = \frac{1}{2}, \\
 W_2 c_2^2 + W_3 c_3^2 + W_4 c_4^2 &= \frac{1}{3}, \quad W_3 c_2 a_{32} + W_4 (c_2 a_{42} + c_3 a_{43}) = \frac{1}{6}, \\
 W_2 c_2^3 + W_3 c_3^3 + W_4 c_4^3 &= \frac{1}{4}, \quad W_3 c_2^2 a_{32} + W_4 (c_2^2 a_{42} + c_3^2 a_{43}) = \frac{1}{12}, \\
 W_3 c_2 c_3 a_{32} + W_4 (c_2 a_{42} + c_3 a_{43}) c_4 &= \frac{1}{8}, \quad W_4 c_2 a_{32} a_{43} = \frac{1}{24}.
 \end{aligned} \tag{14}$$

We have 11 equations in 13 unknowns. Therefore, there are two arbitrary parameters. Since the terms up to $O(h^4)$ are compared, the truncation error is of $O(h^5)$ and the order of the method is 4. The simplest solution of the equations (14) is given by

$$\begin{aligned}
 c_2 = c_3 = \frac{1}{2}, \quad c_4 = 1, \quad W_2 = W_3 = \frac{1}{3}, \\
 W_1 = W_4 = \frac{1}{6}, \quad a_{21} = \frac{1}{2}, \quad a_{31} = 0, \quad a_{32} = \frac{1}{2}, \quad a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = 1.
 \end{aligned}$$

Runge-Kutta method is obtained as

$$y(t_{n+1}) = y(t_n) + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4] \tag{15}$$

where

$$\begin{aligned}
 K_1 &= hf(t_n, y(t_n)) \\
 K_2 &= hf\left(t_n + \frac{h}{2}, y(t_n) + \frac{K_1}{2}\right) \\
 K_3 &= hf\left(t_n + \frac{h}{2}, y(t_n) + \frac{K_2}{2}\right)
 \end{aligned}$$

$$\text{and} \quad K_4 = hf(t_n + h, y(t_n) + K_3).$$

5 Runge-Kutta method of order four for solving Fuzzy Differential

Equations

Let $Y = [\underline{Y}, \bar{Y}]$ be the exact solution and $y = [\underline{y}, \bar{y}]$ be the approximated solution of the fuzzy initial value problem (3).

Let $[Y(t)]_r = [\underline{Y}(t; r), \bar{Y}(t; r)]$, $[y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)]$.

Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$$[Y(t_n)]_r = [\underline{Y}(t_n; r), \bar{Y}(t_n; r)], [y(t_n)]_r = [\underline{y}(t_n; r), \bar{y}(t_n; r)] \quad (0 \leq n \leq N).$$

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, t_i = t_0 + ih, \quad 0 \leq i \leq N.$$

Then we obtain, $\underline{Y}(t_{n+1}; r) = \underline{Y}(t_n; r) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where

$$\begin{aligned} K_1 &= hF \left[t_n, \underline{Y}(t_n; r), \bar{Y}(t_n; r) \right] \\ K_2 &= hF \left[t_n + \frac{h}{2}, \underline{Y}(t_n; r) + \frac{K_1}{2}, \bar{Y}(t_n; r) + \frac{K_1}{2} \right] \\ K_3 &= hF \left[t_n + \frac{h}{2}, \underline{Y}(t_n; r) + \frac{K_2}{2}, \bar{Y}(t_n; r) + \frac{K_2}{2} \right] \\ K_4 &= hF \left[t_n + h, \underline{Y}(t_n; r) + K_3, \bar{Y}(t_n; r) + K_3 \right] \end{aligned} \tag{16}$$

and $\bar{Y}(t_{n+1}; r) = \bar{Y}(t_n; r) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where

$$\begin{aligned} K_1 &= hG \left[t_n, \underline{Y}(t_n; r), \bar{Y}(t_n; r) \right] \\ K_2 &= hG \left[t_n + \frac{h}{2}, \underline{Y}(t_n; r) + \frac{K_1}{2}, \bar{Y}(t_n; r) + \frac{K_1}{2} \right] \\ K_3 &= hG \left[t_n + \frac{h}{2}, \underline{Y}(t_n; r) + \frac{K_2}{2}, \bar{Y}(t_n; r) + \frac{K_2}{2} \right] \\ K_4 &= hG \left[t_n + h, \underline{Y}(t_n; r) + K_3, \bar{Y}(t_n; r) + K_3 \right] \end{aligned} \tag{17}$$

Also we have $\underline{y}(t_{n+1}; r) = \underline{y}(t_n; r) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where

$$\begin{aligned}
 K_1 &= hF \left[t_n, \underline{y}(t_n; r), \bar{y}(t_n; r) \right] \\
 K_2 &= hF \left[t_n + \frac{h}{2}, \underline{y}(t_n; r) + \frac{K_1}{2}, \bar{y}(t_n; r) + \frac{K_1}{2} \right] \\
 K_3 &= hF \left[t_n + \frac{h}{2}, \underline{y}(t_n; r) + \frac{K_2}{2}, \bar{y}(t_n; r) + \frac{K_2}{2} \right] \\
 K_4 &= hF \left[t_n + h, \underline{y}(t_n; r) + K_3, \bar{y}(t_n; r) + K_3 \right]
 \end{aligned} \tag{18}$$

and
$$\bar{y}(t_{n+1}; r) = \bar{y}(t_n; r) + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$

where

$$\begin{aligned}
 K_1 &= hG \left[t_n, \underline{y}(t_n; r), \bar{y}(t_n; r) \right] \\
 K_2 &= hG \left[t_n + \frac{h}{2}, \underline{y}(t_n; r) + \frac{K_1}{2}, \bar{y}(t_n; r) + \frac{K_1}{2} \right] \\
 K_3 &= hG \left[t_n + \frac{h}{2}, \underline{y}(t_n; r) + \frac{K_2}{2}, \bar{y}(t_n; r) + \frac{K_2}{2} \right] \\
 K_4 &= hG \left[t_n + h, \underline{y}(t_n; r) + K_3, \bar{y}(t_n; r) + K_3 \right]
 \end{aligned} \tag{19}$$

Clearly, $\underline{y}(t; r)$ and $\bar{y}(t; r)$ converge to $\underline{Y}(t; r)$ and $\bar{Y}(t; r)$, respectively whenever $h \rightarrow 0$ [4].

6 Numerical Results

In this section, the exact solutions and approximated solutions obtained by Euler's method and Runge-Kutta method of order four are plotted in figure 1 and figure 2.

Example 6.1

Consider the initial value problem

$$\begin{cases}
 y'(t) = f(t), & t \in [0, 1] \\
 y(0) = (0.8 + 0.125r, 1.1 - 0.1r).
 \end{cases}$$

The exact solution at $t = 1$ is given by

$$Y(1; r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e], \quad 0 \leq r \leq 1.$$

Using iterative solution of Runge-Kutta method of order four, we have

$$\underline{y}(0; r) = 0.8 + 0.125r, \quad \bar{y}(0; r) = 1.1 - 0.1r,$$

and by

$$\begin{aligned}\underline{y}^{(0)}(t_{i+1}; r) &= \underline{y}(t_i; r) + h \underline{y}(t_i; r) \\ \bar{y}^{(0)}(t_{i+1}; r) &= \bar{y}(t_i; r) + h \bar{y}(t_i; r),\end{aligned}$$

where $i = 0, 1, \dots, N - 1$ and $h = \frac{I}{N}$. Now, using these equations as an initial guess for following iterative solutions respectively,

$$\underline{y}^j(t_{i+1}; r) = \underline{y}(t_i; r) + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$

where

$$\begin{aligned}K_1 &= h \underline{y}(t_i; r) \\ K_2 &= h \left[\underline{y}(t_i; r) + \frac{K_1}{2} \right] \\ K_3 &= h \left[\underline{y}(t_i; r) + \frac{K_2}{2} \right] \\ K_4 &= h \left[\underline{y}(t_i; r) + K_3 \right]\end{aligned}$$

and
$$\bar{y}^j(t_{i+1}; r) = \bar{y}(t_i; r) + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$

where

$$\begin{aligned}K_1 &= h \bar{y}(t_i; r) \\ K_2 &= h \left[\bar{y}(t_i; r) + \frac{K_1}{2} \right] \\ K_3 &= h \left[\bar{y}(t_i; r) + \frac{K_2}{2} \right] \\ K_4 &= h \left[\bar{y}(t_i; r) + K_3 \right]\end{aligned}$$

and $j = 1, 2, 3$. Thus, we have $\underline{y}(t_i; r) = \underline{y}^{(3)}(t_i; r)$ and $\bar{y}(t_i; r) = \bar{y}^{(3)}(t_i; r)$, for $i = 1 \dots N$.

Therefore, $\underline{Y}(1; r) \approx \underline{y}^{(3)}(1; r)$ and $\bar{Y}(1; r) \approx \bar{y}^{(3)}(1; r)$ are obtained.

Table 3, shows estimation of error for different values of $r \in [0, 1]$ and h .

By minimizing the step size h , the solution by exact method and RK method almost coincides.

r	Exact solution
0	2.174625,2.990110
0.2	2.242583,2.935744
0.4	2.310540,2.881379
0.6	2.378497,2.827013
0.8	2.446454,2.772647
1	2.514411,2.718282

Table 1: Exact solution

h	0.1	0.01
r		
0	1.958475,2.692903	2.174515,2.989958
0.2	2.019677,2.643941	2.242468,2.935595
0.4	2.080880,2.594980	2.310422,2.881232
0.6	2.142082,2.546018	2.378375,2.826869
0.8	2.203284,2.497056	2.446329,2.772506
1	2.264487,2.448094	2.514283,2.718143

Table 2: Approximated solution

h	0.1	0.01
r		
0	0.513357	0.000262
0.2	0.514709	0.000264

0.4	0.516059	0.000265
0.6	0.517410	0.000266
0.8	0.518761	0.000266
1	0.520112	0.000267

Table 3: Error for different values of r and h .

Graphical Representation of exact and approximated solution

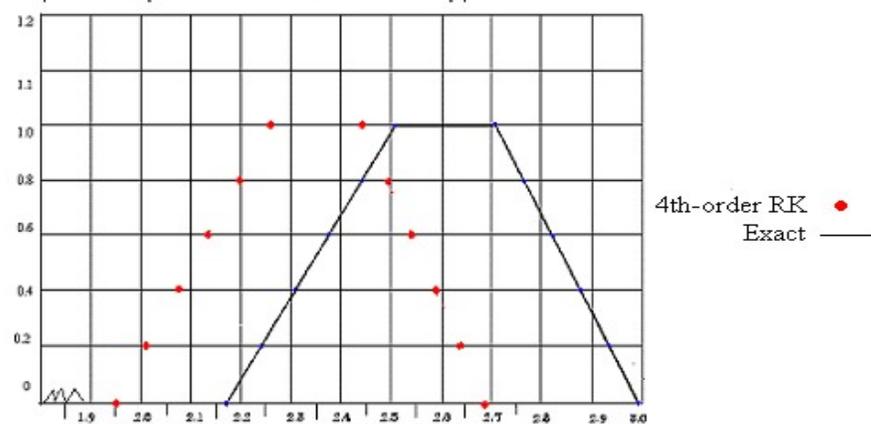


Figure 1 : $h = 0.1$

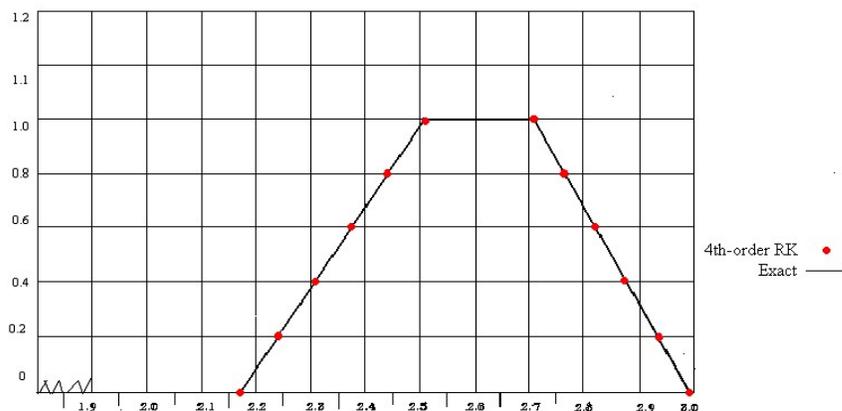
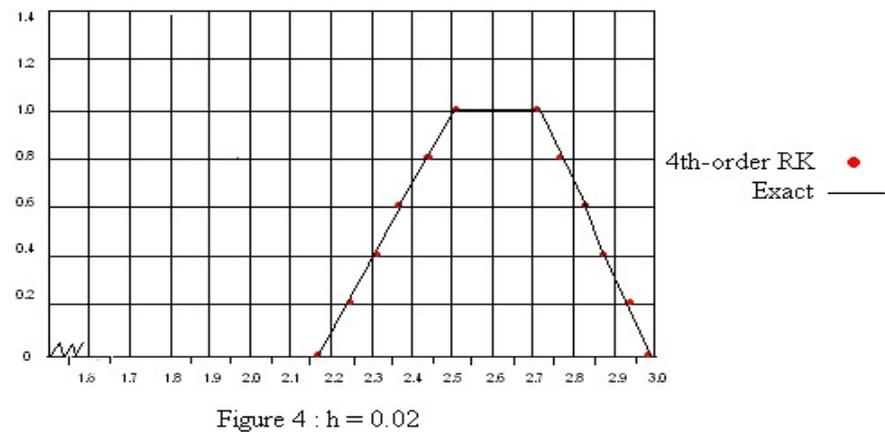
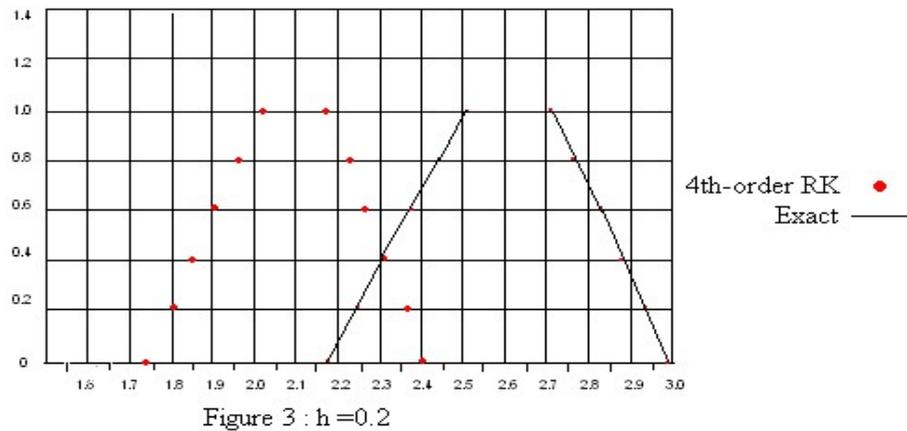


Figure 2 : $h = 0.01$



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